First-order phase transition in the two- and three-dimensional RP^{n+1} and CP^{n-1} models, in the large-n limit

1989 J. Phys. A: Math. Gen. 22 L1043

(http://iopscience.iop.org/0305-4470/22/21/013)

J. Phys. A: Math. Gen. 22 (1989) L1043-L1048. Printed in the UK

LETTER TO THE EDITOR

First-order phase transition in the two- and three-dimensional $\mathbb{R}P^{n+1}$ and $\mathbb{C}P^{n-1}$ models, in the large-n limit

H Kunz and G Zumbach

Institute de Physique Théorique, Ecole Polytechnique Fédéral de Lausanne, PHB-Ecublens, CH-1015 Lausanne, Switzerland

Received 28 July 1989

Abstract. Solving the lattice $\mathbb{R}P^{n-1}$ and $\mathbb{C}P^{n-1}$ models in the large-n limit, we show that they undergo a first-order phase transition for dimensions greater than or equal to two. In the latter case, although there is no long-range order, we argue that at least in the $\mathbb{R}P^{n-1}$ model, the transition corresponds to a condensation of defects.

Non-linear σ models have been thoroughly studied in field theory, especially in the two-dimensional case, because of their analogy with four-dimensional gauge theories. Their vectorial version (XY, Heisenberg) is also quite well known in statistical mechanics. On the other hand, their matrix form, also called chiral or Grassmannian non-linear σ models, is comparatively less well understood. Some of them describe liquid crystals [1, 2], whereas others have been introduced to analyse Anderson localisation [3]. In the three-dimensional case, conflicting conclusions have been reached concerning the nature of the phase transition they undergo. Monte Carlo computations indicate a first-order transition for the $\mathbb{R}P^2$ model [4, 5] or $\mathbb{C}P^3$ and $\mathbb{C}P^4$ models [5], in agreement with a mean-field theory, whereas expansions in the space dimension $4-d$ [6] or $d-2$ [7], or in $1/n$ [8], all consistently give a continuous transition. We have reconsidered the lattice version of the simplest of such models, the $\mathbb{R}P^{n-1}$ and $\mathbb{C}P^{n-1}$ models, in the large-n limit. We find a first-order transition in three dimensions. More surprisingly, this first-order transition still occurs in the two-dimensional case, with vanishing order parameter. However, we argue that this transition should be defect mediated, at least in the $\mathbb{R}P^{n-1}$ case. We also discuss briefly the possible origin of the failure of the $d-2$ expansion.

Consider a d-dimensional lattice of N sites, with periodic boundary conditions. The Hamiltonian of the $\mathbb{R}P^{n-1}$ model is defined by

$$H = -n \sum_{x,\mu} |\boldsymbol{\sigma}(x) \cdot \boldsymbol{\sigma}(x+e_\mu)|^2 \qquad \mu = 1 \ldots d \tag{1}$$

where $\boldsymbol{\sigma}(x)$ is a real n-component vector of unit length, and e_μ is a unit lattice vector in the direction μ. In the $\mathbb{C}P^{n-1}$ model, the Hamiltonian has the same form, but now each vector $\boldsymbol{\sigma}(x)$ has n complex components and

$$\boldsymbol{\sigma}(x)^2 = \sum_{\alpha=1}^{n} \boldsymbol{\sigma}_\alpha(x)^* \boldsymbol{\sigma}_\alpha(x) = 1$$

0305-4470/89/211043+06$02.50

Besides a global symmetry group O(n) (respectively U(n)) in the $\mathbb{R}P^{n-1}$ model (respectively $\mathbb{C}P^{n-1}$), these Hamiltonians are invariant under a local gauge group Z_2 (respectively U(1)) in the $\mathbb{R}P^{n-1}$ case (respectively $\mathbb{C}P^{n-1}$). The $\mathbb{R}P^2$ model is a lattice version of the Meier-Saupe model for the nematic-isotropic transition in liquid crystals, and the $\mathbb{C}P^1$ model is equivalent to the classical Heisenberg model.

Let us now look at the partition function of the $\mathbb{R}P^{n-1}$ model

$$Z = \int \prod_x \mathrm{d}\boldsymbol{\sigma}(x)\delta(\boldsymbol{\sigma}(x)^2-1)\exp\left(\beta n \sum_{x,\mu} (\boldsymbol{\sigma}(x)\cdot\boldsymbol{\sigma}(x+e_\mu))^2\right). \tag{2}$$

Introducing 'gauge fields' $A_\mu(x)$, this can be rewritten as

$$Z = \mathrm{Cte} \int \prod_x \mathrm{d}\boldsymbol{\sigma}(x)\,\delta(x)\,\delta(\boldsymbol{\sigma}(x)^2-1) \int \prod_{x,\mu} \mathrm{d}A_\mu(x)$$
$$\times \exp\left(-\beta n \sum_{x,\mu} (\partial_\mu\boldsymbol{\sigma} + A_\mu\boldsymbol{\sigma})^2(x)\right) \tag{3}$$

where $(\partial_\mu\boldsymbol{\sigma})(x) = \boldsymbol{\sigma}(x+e_\mu) - \boldsymbol{\sigma}(x)$. Using the identity

$$\delta(\boldsymbol{\sigma}^2-1) = \frac{\beta n}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \exp[-\beta n(a+\mathrm{i}t)(\boldsymbol{\sigma}^2-1)] \tag{4}$$

a being an arbitrary real number, we can put it in the useful form

$$Z = \exp[NC((n,\beta))] \int \prod_{x,\leftarrow} \mathrm{d}A_\mu(x)\mathrm{d}t_x \exp\left(-\frac{n}{2}\mathscr{H}(A_\mu, t)\right) \tag{5}$$

where

$$\mathscr{H}(A_\mu, t) = -2\beta \sum_x (a+\mathrm{i}t_x) + \mathrm{Tr}\ln\left(\sum_\mu (\partial_\mu + A_\mu)^{\mathrm{T}}(\partial_\mu + A_\mu) + a + \mathrm{i}t\right) \tag{6}$$

and

$$C(n,\beta) = \beta n d + \tfrac{1}{2}(d-n+2)\ln(\beta n/\pi) - \ln 2. \tag{7}$$

Defining $A_\mu(x) = A + \delta A_\mu(x)$, we then choose A and a such that $\mathscr{H}(A_\mu, t)$ is extremum at $(A, 0)$, ensuring that when n tends to infinity, the partition function will be dominated by the saddle points of the integrand. In the thermodynamic limit, these points can be usefully parametrised by

$$a = \frac{2}{d\Lambda_0^2} - d + \frac{\Lambda^2}{d\Lambda_0^4} \qquad A = 1 \pm \frac{\Lambda}{d\Lambda_0^2} \tag{8}$$

where $\Lambda_0^2 = 4\beta/d$ and Λ is either 0 (trivial solution) or the postive solution of the saddle point equations

$$z = \frac{1}{\Lambda} + \frac{\Lambda}{\Lambda_0^2} \tag{9}$$

$$z = g(\Lambda). \tag{10}$$

We define $g(\Lambda)$ as the inverse of the function

$$f(z) = \int \mathrm{d}\theta \frac{1}{z - \sum_{\mu=1}^{d} \cos\theta_\mu} \tag{11}$$

where $\Lambda < \bar{\Lambda} = f(d)$ and $g(\Lambda) = d$ when $\Lambda \geqslant \bar{\Lambda}$. This last condition comes from the zero mode in the Hamiltonian. We use the notation

$$\int \mathrm{d}\theta = \int_{-\pi}^{\pi} \prod_{\mu} \frac{\mathrm{d}\theta_\mu}{2\pi}. \tag{12}$$

The pressure in the $n = \infty$ limit can be expressed as

$$\beta p_\infty = C(n, \beta) - \beta n d + \tfrac{1}{2} n \ln 2\beta - \tfrac{1}{2} n f \tag{13}$$

with

$$f = \inf_{\{\Lambda\}} \left[-1 - \frac{1}{2} \frac{\Lambda^2}{\Lambda_0^2} + \ln \Lambda + \int \mathrm{d}\theta \ln\left(z - \sum_{\mu=1}^{d} \cos \theta_\mu \right) \right]. \tag{14}$$

In three dimensions or more, an analysis of the saddle point equation shows that when $\Lambda_0 < 2/d$, there is no solution $\Lambda > 0$. When $\Lambda_0 > 2/d$ a non-vanishing solution Λ_+ appears with

$$\frac{\Lambda_+}{\Lambda_0} = \frac{d\Lambda_0}{2} + \left[\left(\frac{d\Lambda_0}{2} \right)^2 - 1 \right]^{1/2} \tag{15}$$

and $z = d$. When $\Lambda_0 = \sqrt{2/d}$, another solution Λ' bifucates from the trivial one $\Lambda = 0$ and for $2/d < \Lambda_0 < \sqrt{2/d}$ these two solutions exist, but the solution Λ_+ has a larger pressure than Λ'. Finally, a comparison of the pressure shows that the trivial solution $\Lambda = 0$ is the stable one until a critical temperature $T_c(\Lambda_0^c > 2/d)$ is reached, above which the solution Λ_+ is the stable one (see figure 1). A similar result holds in two dimensions, where $\bar{\Lambda} = \infty$ (see figure 2). Thus, we see that Λ and $\mathrm{d}\Lambda/\mathrm{d}\beta$ jump at T_c, so that the transition is first order in temperature, with a jump in the internal energy of 1.3 in $d = 3$ and 0.45 in $d = 2$. In one dimension, we recover the continuous transition discussed by Hikami *et al* [9]. When $d \to \infty$, but βd is fixed, we get the mean-field results. For the $\mathbb{C}P^{n-1}$ model, to leading order in n, the results are the same. In fact, we have the following relationship for the pressure of the two models:

$$p(\mathbb{C}P^\infty, \beta) = p(\mathbb{R}P^\infty, \beta/2). \tag{16}$$

For this last model, our results disagree with those obtained for its field theoretical

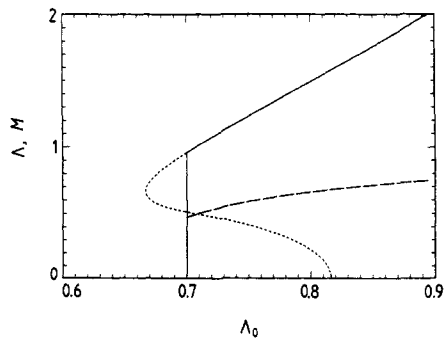

Figure 1. The $d = 3$ case: the full curve (respectively, short broken curve) gives the stable (respectively, unstable) solution Λ and the long broken curve denotes the order parameter M, both as a function of Λ_0.

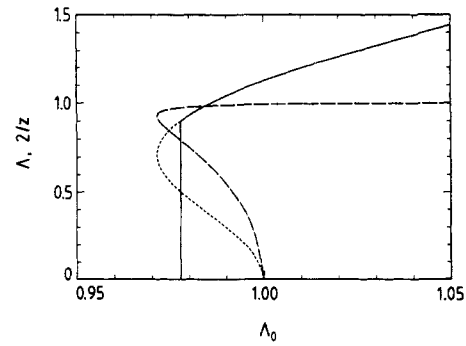

Figure 2. The $d = 2$ case: the full curve (respectively, short broken curve) gives the stable (respectively unstable) solution Λ, and the long broken curve denotes the ratio d/z, both as a function of Λ_0.

point $A=0$, recovering the result of the usual vectorial model in the $n=\infty$ limit. But we have seen that at least in the lattice version, $A=0$ is not a saddle point.

In order to gain further insight into the transition, we can also compute the associated order parameter. The local order parameter $m(x)$ can be chosen as

$$m(x)=\frac{n\sigma_1^2(x)-1}{n-1}. \tag{17}$$

We can compute the correlation functions of these local order parameters, in the absence of a symmetry-breaking field. To leading order in $1/n$, we find that

$$\langle m(x)m(y)\rangle=\frac{2}{n(n-1)}[M+D(x-y)]^2$$

$$M=\begin{cases}0 & \beta<\beta_c\\ 1-\dfrac{f(z)}{\Lambda} & \text{otherwise.}\end{cases} \tag{18}$$

On the other hand

$$D(r)=\begin{cases}\delta_{r,0} & \beta<\beta_c\\ \dfrac{1}{\Lambda}\displaystyle\int \mathrm{d}\theta\frac{\mathrm{e}^{i\theta r}}{z-\sum_{\mu=1}^{d}\cos\theta_\mu} & \beta>\beta_c.\end{cases} \tag{19}$$

Thus we see that in three dimensions or more, we have long-range order below T_c, the average value of the order parameter being given by

$$\langle m\rangle=\left(\frac{2}{n(n-1)}\right)^{1/2}\left(1-\frac{\bar{\Lambda}}{\Lambda_+}\right) \tag{20}$$

and since $z=d$ for $\beta>\beta_c$, the correlation function in the low-temperature phase decays as $1/r^{d-2}$.

From the previous analysis of the saddle point equation, it follows that the order parameter makes a jump at T_c (see figure 1). Its value extrapolated to $n=2$ is 0.29, compared with the Monte Carlo result 0.33 [4]. On the other hand, near two dimensions there is a temperature $T^*=\mathrm{O}(d-2)$ such that for $0<T<T^*$ we have a non-vanishing order parameter, but for $T^*<T<T_c$ the order parameter vanishes. This order parameter vanishes continuously at T^*, with a critical exponent $\beta=1$. This result just corresponds to the prediction of the $d-2$ expansion in the $n\to\infty$ limit. But we think that this is incorrect and as soon as n will be finite, the order parameter will be very small but positive between T^* and T_c because of the phase transition occurring at T_c which is two dimensional in nature. Indeed in two dimensions, equations (9), (10) and (18) give $M=0$, i.e. no long-range order, but an exponential decay of the correlation functions. The correlation length, however, is huge, since $z\simeq d$, as seen in figure 2. This suggests a topological phase transition in two dimensions. To confirm this, we have computed an analogue of the Wilson loop, namely

$$C(L)=\left\langle\prod_{(x,x+e_\mu)\in L}\boldsymbol{\sigma}(x)\cdot\boldsymbol{\sigma}(x+e_\mu)\right\rangle \tag{21}$$

when L is a loop on the lattice. We find that when $n=\infty$

$$C(L)=\begin{cases}0 & \beta<\beta_c\\ \mathrm{e}^{-\gamma|K|} & \beta>\beta_c\end{cases} \tag{22}$$

where $|L|$ is the perimeter of the loop and

$$\gamma=\ln d\Lambda_0^2/\Lambda'. \tag{23}$$

The change from an area-law decay (in the strong form when $n=\infty$) at high temperatures to a perimeter law is indicative of a topological phase transition. All these results remain valid for the $\mathbb{C}P^{\infty}$ model. What is, however, the physical mechanism behind this transition? We think that it is a condensation of defects like in the Kosterlitz-Thouless transition, at least for $\mathbb{R}P^{n-1}$. Indeed, such models possess topologically stable point defects in two dimensions. This comes from the fact that $\Pi_1(\mathbb{R}P^n)=Z_2$ if $n\geqslant 2$. This means that, contrary to vortices, there are only defects of charge 1, but that two defects of charge 1 can annihilate at least when $n\geqslant 3$ (since $\Pi_2(\mathbb{R}P^n)=0$ when $n\geqslant 3$). In the liquid crystal case ($n=3$), these defects correspond to disclinations of number $\frac{1}{2}$, in the standard terminology [12]. Note, however, that, in this case, we should also have textures (or instantons) since $\Pi_2(\mathbb{R}P^2)=Z$. The energy of an isolated defect grows logarithmically with the size of the system and two defects attract each other with a logarithmic potential. If we use, therefore, the familiar energy-entropy balance argument [13], we conclude that the low-temperature phase should be a condensate of pairs of such defects, which should remain isolated at high temperatures. A more detailed theory along the lines sketched here would be needed to analyse the nature of this transition; our results suggest that it is of first order. There is, however, one essential difference between the $\mathbb{R}P^{n-1}$ and $\mathbb{C}P^{n-1}$ manifolds. The $\mathbb{C}P^{n-1}$ models do not possess topologically stable point defects ($\Pi_1(\mathbb{C}P^n)=0$), but only textures, called instantons in field theory ($\Pi_2(\mathbb{C}P^n)=Z$), so that the physical origin of the two-dimensional transition when n is large, remains mysterious to us in this case. The failure of the $d-2$ expansion to detect such transitions is related to the fact that, in this approach, local coordinates are used on the sphere, which do not see that in the projective models opposite points on the sphere have to be identified. Finally, one can wonder if these first-order transitions remain stable when we consider the first correction in $1/n$ to the pressure. This is what we have done for the $\mathbb{R}P^{n-1}$ case. We find to this order that

$$\beta p=\beta p_{\infty}-\frac{1}{2}\int \mathrm{d}\theta[\ln \Gamma(\theta)+\mathrm{Tr}\ln R(\theta)] \tag{24}$$

where $R_{\mu\nu}(\theta)$ is the matrix

$$R_{\mu\nu}(\theta)=\frac{1}{\beta}\delta_{\mu\nu}-\Gamma_{\mu\nu}(\theta)+\frac{\Gamma_{\mu}(\theta)\Gamma_{\nu}(\theta)}{\Gamma(\theta)} \tag{25}$$

with

$$\begin{aligned}
&\Gamma_{\mu\nu}(\theta)=\int \mathrm{d}\theta'\, L(\theta')\beta_{\mu}(\beta_{\nu}^{*}(\theta')\\
&\Gamma_{\mu}(\theta)=\int \mathrm{d}\theta'\, L(\theta')\beta_{\mu}(\theta')\\
&\Gamma(\theta)=\int \mathrm{d}\theta'\, L(\theta')\\
&\beta_{\mu}(\theta')=(\exp(\mathrm{i}\theta'_{\mu})-1)+\exp(-\mathrm{i}\theta_{\mu})(\exp(-\theta'_{\mu})-1)\\
&L(\theta')=\frac{1}{\Lambda^2[z-\Sigma_{\mu}\cos\theta'_{\mu}][z-\Sigma_{\mu}\cos(\theta'_{\mu}+\theta_{\mu})]}.
\end{aligned} \tag{26}$$

These corrections do not introduce, however, a qualitative change in the transition.

We would like to thank A Sütő, Ch Pfister and Ph Cohoquard for useful discussions. This work was supported by the Fonds National Suisse de la Recherche Scientific.

References

[1] Lubensky T C and Priest R G 1974 *Phys. Lett.* **48A** 103-104
[2] Priest R G and Lubensky T C 1976 *Phys. Rev.* B **13** 4159-71
[3] Shaefer L and Wegner F 1980 *Z. Phys.* B **38** 113-26
[4] Lebwohl P A and Lasher G 1972 *Phys. Rev.* A **6** 426-28; 1973 *Phys. Rev.* A **7** 2222
[5] Duane S and Green M B 1981 *Phys. Lett.* **103B** 359-64
[6] Hikami S 1980 *Prog. Theor. Phys.* **64** 1425-32
[7] Brézin E, Hikami S and Zinn-Justin J *Nucl. Phys.* B **165** 528-44
[8] Vasili'ev A N, Nalimov N Yu and Khonkonen Yu R 1984 *Theor. Math. Phys.* **58** 11-20
[9] Hikami S and Maskawa T 1982 *Prog. Theor. Phys.* **67** 1038-52
[10] D'Adda A, Luscher M and Vecchia P Di 1978 *Nucl. Phys.* B **146** 63-76
[11] Witten E 1979 *Nucl. Phys.* B **149** 285-320
[12] de Gennes P G 1974 *The Physics of Liquid Crystals* (Oxford: Clarendon)
[13] Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181-203